CULTIVATING AN AREA MODEL
Students often have difficulty using mathematical notation to symbolize relationships among quantities (Knuth et al. 2006). This difficulty becomes especially problematic during algebra instruction when the purpose of solving problems is to identify and symbolize relationships. For example, if a quantity \((a + b)\) is to be squared, then a common goal is for students to understand that \((a + b)^2\) is \(a^2 + ab + ab + b^2\), or \(a^2 + 2ab + b^2\).

Many students are not interested in these types of equivalences, partly because learning them becomes simply trying to remember a rule. For instance, students often conclude that the common binomial \((a + b)^2\) can be written as \(a^2 + b^2\). They base their solution on an overgeneralization of this memorized rule: “You move the 2 inside the parenthesis,” using the distributive property in an expression such as \(2(a + b) = 2a + 2b\). When students draw such conclusions, it suggests that they may be relying on memorized rules rather than using meaningful mathematical reasoning.

This article describes how to multiply binomials so that middle school students produce the equivalence \((a + b)^2 = a^2 + 2ab + b^2\) after reasoning about and representing problems. First, a two-dimensional array representation is used so that students can become comfortable with multiplication embedded in their figures. Later, an area model is used. Such area models provide multiple representations of squaring a binomial; in so doing, these diagrams give students multiple ways to understand the concept.

I have used the problems here, and variations of them, with eighth-grade prealgebra and algebra 1 students, as well as with preservice middle school teachers. Each problem can take at least a week of class time, although the time can be adjusted, depending on a teacher’s goals. For algebra 1 students, the problems provide a basis for future work with quadratic equations and functions.

In general, the problems help students produce meaningful algebraic notation for multiplying binomials by having the notation result from “naming” or “viewing” quantities in different ways. When working with these tasks, students find that what they thought was a mundane mathematical rule becomes an interesting way of thinking.

**A COMMON PROBLEM**
Producing this relationship,

\[
\text{if } c = (a + b), \text{ then } c^2 = (a + b)^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2,
\]

is quite difficult for students. The Backyard problem explores why students struggle.

**The Backyard Problem**
A man has a square backyard. The length of each side of the backyard is 10 meters plus an unknown length. Make a picture of the backyard, and find a symbolic expression for the area.

One student represented the area correctly in his solution illustration (see fig. 1); however, the symbolic expression is incorrect. This particular error in symbolizing the problem occurs frequently. In this case, the exponent implies that there are four \((10 + x)\) lengths in the drawing. Because these difficulties are common, I developed the approach described in this article.

**ARRAY REPRESENTATIONS**
The approach begins with a presentation of problems that involve pairing, such as the Outfits problem.

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**Fig. 1** The student is able to correctly represent the Backyard problem using a drawing; however, the symbolic representation incorrectly uses 4 as an exponent rather than a coefficient.
The Outfits Problem
An outfit is a combination of a shirt and a pair of pants. How many outfits could you make with four shirts and three pairs of pants?

These types of problems are presented with two goals in mind: to help students (1) understand that these problems involve multiplication and (2) produce an array to represent their reasoning. Some students may order the shirts and pants by color, letter, or number, then list each possible outfit and count the total number one by one (fig. 2). Counting the elements one by one usually means that the student is not considering multiplication.

Once students understand that the situation involves multiplication, which may take one to two class periods, they can be asked to represent a problem using an array, as in figure 3. An array shows that each point can be found only by choosing a particular pair of pants and a particular shirt (e.g., the red shirt and red pants are one point). This locating-of-points method resembles the Cartesian coordinate system (i.e., to locate a point, both an x and y coordinate are needed). This skill is important for work in algebra.

THE BINOMIAL PATTERN IN ARRAYS
Once they are comfortable producing arrays, students can be asked to deal with the Card problem.

The Card Problem
You have the ace through the king of hearts (13 cards). Your friend has the ace through the king of clubs (13 cards). Use an array to show all possible two-card hands you could make that consist of 1 heart and 1 club. On your array, show how many two-card hands have—

1. 2 face cards (the face cards are the jack, queen, and king),

2. 1 face card, and

3. no face cards.

Use these sections of your array to determine how many total two-card hands you can make.

To solve this problem, students often think about putting the ace of hearts with all 13 clubs to make the first 13 two-card hands; they then repeat this process with the remaining 12 hearts. To represent their solution to this problem, students can be asked to use an array and then break the array into four sections to locate the two-card hands with 2, 1, and no face cards (see fig. 4). This problem is usually difficult for students, so teachers should be prepared to ask an appropriate question such as this: How could you arrange your array so that there will be only four sections when you break it apart?

Over one or two days, similar types of problems can be presented, such as the Password problem.

The Password Problem
You have to create a two-character password for your computer. You can use the digits 0–9 and the letters A–D. Represent all possible passwords using an array. On your array, show the number of passwords that have 2 numbers, that have exactly 1 number and 1 letter, and that have 2 letters. Use these sections of your array to determine the total number of passwords.

The Card and Password problems involve similar reasoning; the array for the Password problem is similar to that shown in figure 4. When students have made these two arrays,
they can be asked to look for patterns in their solution of the two problems. Students usually respond that—

- each array has four sections.
- both arrays have two squares and two identical rectangles.
- both arrays have a section that represents 100 possibilities.

As students start to notice similarities between the two problems, teachers can ask them to use these patterns to find a way to square any teen number (any number from 13 to 19) mentally. One way is to present a problem similar to the Card and Password problems. However, ask students to predict what they think the final array would look like without actually making it. They can use their prediction to calculate the total number of possible pairings (i.e., two-card hands or passwords). This work continues until students are able to square any teen number mentally.

THE SYMBOLS IN THESE CONTEXTS
In concert with this work, students can begin to use numerical representations in their reasoning. It will be meaningful to them, because the representations refer to different ways of seeing the quantities in the problem. For example, the Card problem’s $13^2$ refers to the total number of two-card hands that can be made; $(10 \times 3)^2$ refers to this same total after the hearts and clubs have been broken into nonface and face cards. Similarly, each of the following—$10^2$, $(10 \times 3)$, $(3 \times 10)$, and $3^2$—refer to parts of the array, specifically, parts of the model counting pairs with 0 face cards, 1 face card, or 2 face cards. Many students will develop the following equivalence:

$$10^2 + (10 \times 3) + (3 \times 10) + 3^2 = 10^2 + 2(10 \times 3) + 3^2$$

Combining the two $(10 \times 3)$ parts of the array is supported both by the representation of the situation as an array and by the problem situation. That is, the numeric expressions $(10 \times 3)$ and $(3 \times 10)$ refer to two congruent parts of the array; each of these parts of the array represent two-card hands that have exactly 1 face card.

THE BINOMIAL PATTERN FOR TWO-DIGIT SQUARE NUMBERS
When students are comfortable squaring teen numbers mentally and are able to use mathematical symbols to express the appropriate relationships, help them square decade numbers (e.g., 10, 20, 30, etc.). With larger numbers, students can represent the problems using dot paper or various computer programs, such as JavaBars (Biddlecomb and Olive 2000) and The Geometer’s Sketchpad (Jackiw 2001). Again, these problems take one to two days. The Farmer’s Field problem can help students learn to square decade numbers.

The Farmer’s Field
A farmer has a field that is 20 dkm by 20 dkm. He can plant a different crop in a 10 dkm by 10 dkm section. Can you show how many different crops he could plant and use that number to figure out the total area of his field?

To begin, ask students to use a drawing to represent the problem (see fig. 5a). Once students have a drawing, they can use it to reason in many different ways. These differences are often evident in the numerical representations that they produce. For example, some students begin simply by suggesting that the farmer could plant four different crops and that each crop would take up 100 dkm. They might express this observation as $20^2 = 4 \times 100 = 400$. Other students may think about each side length of the field being $(10 + 10)$ in a similar way to the problems they
have solved with teen numbers. In that case, they might then express the area in this way:

\[ 20^2 = (10 + 10)^2 = 10^2 + 10^2 = 4 \times (10^2) = 400 \]

Others may break the length of each side in half and think about 20 as \((2 \times 10)\). This observation can lead to the series of equivalences

\[ 20^2 = (2 \times 10)^2 = 2^2 \times 10^2 = 400. \]

To help students become comfortable, encourage them to find as many different ways of thinking about and expressing the situation as possible.

Once students have solved the Farmer’s Field problem, a similar problem finding the number of 10 dkm by 10 dkm areas in a 30 dkm by 30 dkm field is posed. Again, students can be asked to represent this problem using a drawing. Once these two problems have been solved, ask students to predict the number of 10 dkm by 10 dkm areas that will fit in a 40 dkm by 40 dkm field or a 50 dkm by 50 dkm field. Some students may be able to predict this immediately; others may need to continue making drawings.

As students gain experience squaring decade numbers, encourage them to combine the varied reasoning they have used. The Playground problem will draw on that reasoning.

**The Playground Problem**

A playground is 27 m by 27 m. Break the playground into four parts to find the total area.

One solution is to break the playground into a 20 m by 20 m square, a 7 m by 7 m square, and two 20 m by 7 m rectangles (see Fig. 5b). In solving this problem, students should be asked to use what they have already learned. If a student breaks the square (as shown in Fig. 5b), ask him or her to figure the total number of 10 m by 10 m squares that will fit in a 20 m by 20 m square. A question such as this supports the purpose of the problems: to help students develop relationships among different quantities and describe these relationships.

This purpose is different from simply asking students to calculate the total area using a formula. Questions should focus on a particular relationship (i.e., How many 10 m by 10 m squares are contained in a 20 m by 20 m square?) instead of simply knowing and appropriately using the area formula.

**A COMMON PROBLEM RESOLVED**

After solving problems such as those in this article, the student who incorrectly symbolized the Backyard problem in figure 1 produced the symbolic expression for the following Square problem.

**The Square Problem**

A square has a side length of \((A + B)\). Find two ways to express the area of the square.

The student wrote the following:

\[ (A + B)^2 = (A \cdot B) \cdot 2 + A^2 + B^2 \]

In this equation, the left-hand side expresses the total area as the length \((A + B)\) times the width, which is also
A + B), and the right-hand side refers to the sum of the four subareas. The student then used his two ways of expressing the area to compute how to square 17 (see his symbolic solution in fig. 6).

EXTENSIONS
Once students have become experienced with this type of reasoning, unknowns are introduced. An unknown is treated as a quantity that can be, but has not been, measured. For example, students know that any field has a length, although there are many fields that they have not measured. In these contexts, a student might have a field in mind that is (x + 3) dkm by (x + 3) dkm, where x dkm symbolizes a length that has yet to be measured; x + 3 dkm, a length that is 3 dkm longer than the unknown length; and (x + 3)² dkm, an area that can be broken into four subregions (see fig. 7). In this context, students can produce the equivalence,

\[(x + 3)^2 = x^2 + 3x + 3x + 9 = x^2 + 6x + 9,\]

as an extension of the reasoning they used in the problems described earlier.

In situations that involve unknowns, teachers can continue to refer to the algebraic notation in close concert with the quantities they describe. For example, a student can be asked to compute the number of rectangular regions in his or her drawing that have an area of 3x and show this number using algebraic notation (e.g., 2 • (3x)). Ultimately, these types of problems, coupled with good teacher questioning, help students use algebraic notation in ways that will be meaningful to them. The problems will also help students begin to view algebra as a way to symbolize relationships among quantities, not just a field of study requiring rule memorization.

REFERENCES

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