Consider a coalition of agents involved in a joint activity, each endowed with a nonnegative input. Through the activity, the aggregate input of the coalition outputs a total cost. The cost sharing problem is then concerned with how to divide the total cost among the participating agents.

The problem in the abstract form above has long been studied in economics in the context of sharing production costs of a single divisible homogeneous good. Our motivation, on the other hand, comes from the risk sharing problem in the insurance literature and the inventory pooling problem in the operations research literature. We take the former as an example. In this case, each agent, representing an insurance company, is endowed with a random claim, whose risk is measured by the standard deviation. Suppose the random claims of the agents are independent. By pooling the claims, the risk of the total claim becomes square root of the total variance. The risk sharing problem in this special case can be cast as a cost sharing problem. The general version of the problem and its historical origin can be found in, for instance, Ruschendorf (2013). Similarly, cost sharing problem also emerges as special cases of inventory pooling problem and we refer readers to Muller et al. (2002), Chen and Zhang (2009), Chen et al. (2017) for detailed description of the problem.

Numerous solution concepts have been proposed or applied to allocate the cost, ranging from the simple and easily implementable average cost pricing rule to the sophisticated and computationally demanding Shapley value. As its name suggests, average cost pricing rule uses the average cost as a price that uniformly applies to all agents with agent i's cost being her input times the price. It arises commonly in the analysis of various applications in the operations research literature. For example, it is known as the proportional allocation rule in Meca et al. (2004) who study the allocation problem of joint ordering with economy of scale,
and in Ozen et al. (2011) who examine allocations in queueing systems. It also coincides with the so-called dual-based allocation rule, a construction based on duality theory in optimization, in some special cases of the risk sharing problem (Chen et al. 2017) and inventory pooling problem (Chen and Zhang 2009).

While the average cost pricing rule is specifically proposed for the cost sharing problem, the Shapley value, originated from cooperative game theory, applies to cooperative situations beyond cost sharing problem. Indeed, the Shapley value allocates according to the marginal contribution made by agent $i$ when joining coalition $S$, averaged over all possible coalitions. Its definition depends on input only implicitly through the cost function and is in general well defined as long as the cost of the coalition $S$ is defined. The Shapley value is commonly believed to be an "equitable" allocation (Champsaur 1975) and there is also a huge amount of works devoted to the axioms and properties of the Shapley value. Among them, Moulin and Shenker (1994), Moulin (1996), and Sudholter (1998) have specifically made a comparison between the Shapley value and average cost pricing in the cost sharing problem. While they share a common pool of desirable properties, each has its own limitations as well—a certain property holds for one but not the other.

In this paper, instead of creating a dichotomy between the two, we bridge the two allocations by ordering them along the dimension of allocation inequality, a concept dating back to Lorenz (1905), who proposed the partial order—now known as the Lorenz order—to measure the income inequality in a society. In the cost sharing problem, we use Lorenz order or the equivalent concept of majorization (Marshall et al. 2011) to characterize inequality in the allocation of the total cost. Interestingly, the order between average cost pricing and the Shapley value depends on the convexity or concavity of the marginal cost. The common emphasis in the cost sharing literature is on the convexity of the cost function itself (e.g., Moulin and Shenker 1992, Moulin 1996, and de Frutos 1998). To the best of our knowledge,
the convexity of the marginal cost is not brought up before. Our main result states that when the marginal cost is convex, average cost pricing majorizes the Shapley value (or the Shapley value Lorenz dominates average cost pricing). In other words, the Shapley value allocates cost more equally than average cost pricing. The order is reversed when the marginal cost is concave. This result should be viewed in light of the fact that Lorenz order is only a partial order and it is quite possible that the two allocations cannot be ordered at all. Indeed, we provide examples showing that neither one majorizes the other when the marginal cost is neither convex nor concave. Our proof, built on establishing Lorenz order over certain input vectors, maybe of independent interest.

References