Preservation of Additive Convexity and Its Applications in Stochastic Optimization Problems

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Stochastic optimization problems in which the decision maker has to make sequential decisions in the presence of uncertainties are a wide class of problems and have tremendous applications in operations research and management. Dynamic programming pioneered by Richard E. Bellman is without any doubt the most influential approach to study such problems and derive the optimal policies. As dynamic programming divides stochastic optimization problems into a sequence of subproblems and finds an optimal solution by recursively solving all the subproblems, it is usually intractable, albeit the rapid advances on computing technologies, for many practical problems with large state spaces. This major challenge of dynamic programming is well-known as “the curse of dimensionality”, i.e., the size of the state space explodes exponentially in the dimension of the state variables. Therefore, researchers usually have to resort to suboptimal heuristic policies for complex stochastic optimization problems with multiple state variables.

Clark and Scarf (1960) is one of the most celebrated and influential work in operations management. In this classic paper, the authors studied a fundamental stochastic serial inventory system and formulated the optimization problem as a high-dimensional dynamic program. By introducing the concept of echelon inventory, they showed that the multi-variate value functions are additively convex in the echelon inventory levels. Here, we call a function $f(\cdot)$, defined on a convex set $V$ in $\mathbb{R}^n$, to be additively convex if $f(x) = \sum_{k=1}^{n} f_k(x_k)$ for any $x := (x_1, \ldots, x_n) \in V$, where $f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot)$ are univariate convex functions. This key structural property leads to the optimality of the echelon base-stock policy and reduces the calculation of multi-variate value functions into the calculations.
of multiple univariate convex functions, which fundamentally reduces the computation complexity and avoids the curse of dimensionality. Since this pioneering work, researchers have identified several other multi-stage inventory systems with additively convex value functions, including assembly systems (Rosling, 1989; Angelus and Özer, 2016) and serial systems with additional features such as expediting (Lawson and Porteus, 2000), advance demand information (Gallego and Özer, 2003), two stages with finite capacities (Parker and Kapuscinski, 2004), stochastic lead times (Kim et al., 2015), cash consideration (Luo and Shang, 2015), and short-term take-or-pay contract (Goh and Porteus, 2016). These systems constitute an important class of complex stochastic optimization problems for which dynamic programming does not suffer from the curse of dimensionality and can be used to calculate the optimal policies efficiently.

In this paper, we establish two new preservation results of additive convexity for another different class of stochastic optimization problems with additively convex value functions. These two results correspond to a class of optimal transformation problems and a class of optimal disposal problems, respectively. For both classes of problems, there are multiple resources and the optimal policies provide different priorities to transform/dispose these resources (i.e., it is optimal to transform/dispose a resource of a lower priority only after all the resources of higher priorities are transformed/disposed). The priority properties of the optimal policies follow from the monotonicity properties of the objective functions of the optimization problems and are crucial in establishing our preservation results. This salient feature also distinguishes our problems from the multi-stage inventory systems mentioned above (where no priority exists in the optimal policies). We demonstrate the applications of our preservation results to three important stochastic optimization problems in operations management.

The first application is the stochastic inventory management problem with remanufacturing. Zhou et al. (2011) formulated this problem as a high-dimensional dynamic program and proved that the value functions (with properly defined state variables) are additively convex by exploiting the priority among the optimal manufacturing and remanufacturing decisions and the priority among the optimal disposal decisions. Their proof relies on full characterization of the optimal policy; and it is very complex and tedious even for the case with only two types of returns. In §3, we revisit this problem and apply our preservation results to re-establish the additive-convexity result with a new and much simpler proof. We also remark in §3 that our preservation results can be applied to (re-)establish the additive-convexity results for several other stochastic inventory management problems with remanufacturing.

The second application is the dynamic inventory rationing problem with backlogging and multiple demand classes. Topkis first formulated this problem as a high-dimensional dynamic program
and showed the optimality of a rationing level policy by exploiting the priority among the optimal demand fulfillment decisions. He also raised the difficulty of calculating the optimal rationing levels due to the recursive tabulation of functions of several variables (i.e., the curse of dimensionality). Recently, Bao et al. (2017) revisited Topkis (1968)’s model and proved that after a novel state transformation the value functions are additively convex. Hence, the curse of dimensionality does not arise in this problem and the optimal rationing levels can be calculated efficiently. Nevertheless, their proof also relies on full characterization of the optimal policy and it is very complex. In §4, we revisit Topkis (1968)’s model and apply our preservation results to prove the additive-convexity result in a straightforward and much simpler way.

The third application is the dynamic capacity management problem with general upgrading Yu et al. (2015) formulated it as a high-dimensional dynamic program and established the optimality of a parallel and sequential rationing (PSR) policy by exploiting the priority among the optimal demand fulfillment and upgrading decisions. Since the optimal PSR policy is intractable due to the curse of dimensionality, the authors proposed an effective heuristic policy. In §5, we revisit this problem and apply our preservation results to establish that the value functions for a special case of this problem can be decomposed as the sum of univariate concave functions, and hence the optimal PSR policy can be calculated efficiently. Furthermore, these results motivate us to develop an insightful heuristic policy for the general problem and we show through a numerical study that it performs consistently very close to optimal.

Besides the above problems, there are many other stochastic optimization problems where priority exists among the optimal decisions. We expect our preservation results to be useful in identifying more stochastic optimization problems with priority structures and additively convex value functions. We believe that these problems would constitute another important class of complex stochastic optimization problems where dynamic programming does not suffer from the curse of dimensionality and can be used to calculate the optimal policies efficiently.