Stack representation for pretopoi: Toward logical schemes

The Legacy of Jim Lambek
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Background

As a student, I was inspired by a paper of Lambek’s (with I. Moerdijk), “Two sheaf representations for topoi”, which applied algebra to logic via category theory. The main result of my own thesis was an extension of this idea.

The basic model for these results was Grothendieck’s sheaf representation for commutative rings. In subsequent work with two PhD students we have pursued this analogy further:

With H. Forssell, we developed the “site” for the sheaf representation of a boolean pretopos as the topological groupoid of models, resulting in a Stone duality for first-order logic.

With S. Breiner, we added the “structure sheaf” of local pretopoi, to arrive the notion of a “logical scheme”, which combines the syntax and semantics of a logical theory into a single object with both aspects.
Outline: Several sheaf representations

- Commutative rings: Grothendieck
- Toposes 1: Lambek and Moerdijk
- Toposes 2: Lambek
- Toposes 3: Awodey
- Boolean algebras: Stone duality
- Pretoposes 1: A. and Forssell
- Pretoposes 2: A. and Breiner
Grothendieck sheaf representation for commutative rings

Definition
A ring (commutative, with unit $1 \neq 0$) is called local if it has a unique maximal ideal. Equivalently:

$$x + y = 1 \implies x = 1 \text{ or } y = 1.$$  

Theorem (Grothendieck)
Let $A$ be a ring. There is a space $X$ with a sheaf of rings $\mathcal{O}$ such that:

1. for every $p \in X$, the stalk $\mathcal{O}_p$ is a local ring,
2. for the ring of global sections, we have: $\Gamma(\mathcal{O}) \cong A$.

Thus every ring is isomorphic to the ring of global sections of a sheaf of local rings.
Grothendieck sheaf representation for commutative rings

The **space** $X$ is the *prime spectrum* $\text{Spec}(A)$:

1. points $p \in \text{Spec}(A)$ are prime ideals $p \subseteq A$,
2. the topology has basic opens of the following form, for all $f \in A$:
   
   $$B_f = \{ p \in \text{Spec}(A) \mid f \notin p \}.$$

The **structure sheaf** $\mathcal{O}$ is determined by “localizing” $A$ at $f$,

$$\mathcal{O}(B_f) = [f]^{-1}A$$

where $A \to [f]^{-1}A$ freely inverts all of the elements $f, f^2, f^3, \ldots$.

The **stalk** $\mathcal{O}_p$ is then the localization of $A$ at $p$,

$$\mathcal{O}_p = S^{-1}A,$$

where $S = A \setminus p$. 

Grothendieck sheaf representation for commutative rings

The **affine scheme** \((\text{Spec}(A), \mathcal{O})\) represents \(A\) as a “ring of continuous functions”

\[
f : \text{Spec}(A) \longrightarrow \mathcal{O},
\]

except that the ring \(\mathcal{O}\) is itself “varying continuously over the space \(\text{Spec}(A)\)” (i.e. it is a sheaf).

The local ring \(\mathcal{O}_p\) has a **unique maximal ideal**, consisting of “those functions \(f : \text{Spec}(A) \longrightarrow \mathcal{O}\) that vanish at \(p\)”.

It is a “representation” of \(A\) because there is always an injective homomorphism

\[
A \cong \Gamma(\mathcal{O}) \hookrightarrow \prod_p \mathcal{O}_p.
\]

**Corollary (Sub-direct-product representation)**

*Every ring \(A\) is isomorphic to a subring of a direct product of local rings.*
Lambek-Moerdijk sheaf representation for toposes

Definition
A (small, elementary) topos is called sublocal if its subterminal lattice \( \text{Sub}(1) \) has a unique maximal ideal. Equivalently, for \( x, y \in \text{Sub}(1) \):

\[
x \lor y = 1 \quad \text{implies} \quad x = 1 \text{ or } y = 1.
\]

Theorem (Lambek-Moerdijk 1982)
Let \( \mathcal{E} \) be a topos. There is a space \( X \) with a sheaf of topoi \( \tilde{\mathcal{E}} \) such that:

1. for every \( p \in X \), the stalk \( \tilde{\mathcal{E}}_p \) is a sublocal topos,
2. for the topos of global sections, we have: \( \Gamma(\tilde{\mathcal{E}}) \cong \mathcal{E} \).

Thus every topos is isomorphic to the topos of global sections of a sheaf of sublocal topoi.
The space $X$ is the so-called (sub)spectrum of the topos, $\text{Spec}(\mathcal{E})$. It is the prime spectrum of the distributive lattice $\text{Sub}(1)$:

1. the points $P \in \text{Spec}(\mathcal{E})$ are prime ideals $P \subseteq \text{Sub}(1)$,
2. the basic opens have the following form, for all $q \in \text{Sub}(1)$:

$$B_q = \{ P \in \text{Spec}(\mathcal{E}) \mid q \not\in P \}.$$

The lattice of all open sets of $\text{Spec}(\mathcal{E})$ is isomorphic to the ideal completion of $\text{Sub}(1)$,

$$O(\text{Spec}(\mathcal{E})) = \text{Idl(\text{Sub}(1))}.$$
The structure sheaf $\tilde{\mathcal{E}}$ is determined by “slicing” $\mathcal{E}$ at $q \in \text{Sub}(1)$,

$$\tilde{\mathcal{E}}(B_q) = \mathcal{E}/q.$$ 

This takes the place of localization. Note that it also “inverts” all those elements $p \in \text{Sub}(1)$ with $q \leq p$.

For the global sections $\Gamma$, we have:

$$\Gamma(\tilde{\mathcal{E}}) \cong \tilde{\mathcal{E}}(B_\top) = \mathcal{E}/1 \cong \mathcal{E}.$$ 

So the topos of global sections of $\tilde{\mathcal{E}}$ is indeed isomorphic to $\mathcal{E}$.
The stalk $\tilde{\mathcal{E}}_P$ at a prime ideal $P \in \text{Spec}(\mathcal{E})$ is the filter-quotient topos,

$$\tilde{\mathcal{E}}_P = \lim_{\substack{\to \quad q \not\in P}} \mathcal{E}/q,$$

at the prime filter $\text{Sub}(1) \setminus P$.

One then has:

$$\text{Sub}_{\tilde{\mathcal{E}}_P}(1) \cong P,$$

so the stalk topos $\tilde{\mathcal{E}}_P$ is indeed sublocal.
Lambek-Moerdijk sheaf representation for toposes

Again, there is always an injection from the global sections into the product of the stalks,

$$\mathcal{E} \cong \Gamma(\tilde{\mathcal{E}}) \hookrightarrow \prod_{P \in X} \tilde{\mathcal{E}}_P.$$  

Corollary (Sub-direct-product representation for toposes)

*Every topos $\mathcal{E}$ is isomorphic to a subtopos of a direct product of sublocal toposes.*
We have the following **logical interpretation** of the sheaf representation:

- A topos $\mathcal{E}$ is (the term model of) a theory in Intuitionistic Higher-Order Logic.
- A sublocal topos $\mathcal{S}$ is one that has the disjunction property:
  \[
  \mathcal{S} \vdash p \lor q \quad \text{iff} \quad \mathcal{S} \vdash p \quad \text{or} \quad \mathcal{S} \vdash q,
  \]
  for all “propositions” $p, q$.
- The subdirect-product embedding is a logical completeness theorem with respect to such “semantic” toposes $\mathcal{S}$.
- The sheaf representation is a Kripke-style completeness theorem for IHOL, with $\tilde{\mathcal{E}}$ as a “sheaf of possible worlds”.

**Lambek-Moerdijk sheaf representation for toposes**
Lambek’s modified sheaf representation for toposes

But this result is not entirely satisfactory, because we would like the “semantic worlds” $S$ to also have the existence property:

$$S \vdash (\exists x : A) \varphi(x) \iff S \vdash \varphi(a) \text{ for some closed } a : A,$$

(we know that we can prove completeness with respect to such).

**Definition**

A topos $S$ is called **local** if the terminal object 1 is indecomposable and projective, i.e. the global sections functor

$$\Gamma = \text{hom}_S(1, -) : S \rightarrow \text{Set}$$

preserves coproducts and epimorphisms.

Note that a local topos has **both** the disjunction and existence properties.
Lambek’s modified sheaf representation for toposes

Lambek gave the following improvement over the sublocal sheaf representation:

Theorem (Lambek 1989)

Let \( E \) be a topos. There is a faithful logical functor \( E \rightarrowtail F \) and a space \( X \) with a sheaf of topoi \( \tilde{F} \) such that:

1. for every \( p \in X \), the stalk \( \tilde{F}_p \) is a local topos,
2. for the topos of global sections, we have: \( \Gamma(\tilde{F}) \cong F \).

Thus every topos is a subtopos of one that is isomorphic to the topos of global sections of a sheaf of local toposes.

This suffices for a sub-direct-product representation into local toposes, and therefore gives the desired logical completeness with respect to local toposes.

But conceptually it is still not entirely satisfactory.
Local sheaf representation for toposes

In my thesis, I proved:

Theorem (Awodey 1998)

Let $\mathcal{E}$ be a topos. There is a space $X$ with a sheaf of toposes $\tilde{\mathcal{E}}$ such that:

1. for every $p \in X$, the stalk $\tilde{\mathcal{E}}_p$ is a local topos,
2. for the topos of global sections, we have: $\Gamma(\tilde{\mathcal{E}}) \cong \mathcal{E}$.

Thus every topos is isomorphic to the global sections of a sheaf of local toposes.

As before, this gives a sub-direct-product representation,

$$\mathcal{E} \rightarrow \prod_p S_p$$

into a product of local toposes $S_p$, and therefore implies the desired logical completeness of IHOL with respect to local toposes.
Local sheaf representation for toposes

The stronger result also gives better “Kripke semantics” for IHOL, since the “sheaf of possible worlds” now has local stalks.

For classical higher-order logic, more can be said:

**Lemma**

*Every local boolean topos is well-pointed, i.e. the global sections functor,*

\[ \Gamma = \text{hom}_S(1, -) : S \to \text{Set} \]

*is faithful.*

A well-pointed topos is essentially a model of set theory.

**Corollary**

*Every boolean topos is isomorphic to the global sections of a sheaf of well-pointed toposes.*
Local sheaf representation for toposes

For boolean toposes, we therefore have the representation,

\[ \mathcal{B} \rightarrow \prod_{p} S_{p} \]

as sub-direct-product of \textit{well-pointed} toposes \( S_{p} \), along with its logical counterpart:

\textbf{Corollary}

\textit{Classical HOL is complete with respect to models in well-pointed toposes.}

These are \textbf{standard} models of classical HOL, taken in varying ("non-standard") models of set theory.
Local sheaf representation for toposes

Taking the global sections $\Gamma: S_p \rightarrow \text{Set}$ of each such well-pointed model then embeds any boolean topos $\mathcal{B}$ into a power of $\text{Set}$:

$$\mathcal{B} \rightarrow \prod_p S_p \rightarrow \prod_p \text{Set}_p \cong \text{Set}^X,$$

The various composites $\mathcal{B} \rightarrow S_p \rightarrow \text{Set}$ are Henkin style, “non-standard” models of HOL in $\text{Set}$.

**Corollary**

*Classical HOL is complete with respect to Henkin models in Set.*

These Henkin models can be taken as the points of the space $X_\varepsilon$ for the sheaf representation.
Local sheaf representation for toposes

To define the space $X_\mathcal{E}$ of models:

In the sublocal case, the points were prime ideals $p \subseteq \text{Sub}(1)$. These correspond exactly to lattice homomorphisms

$$p : \text{Sub}_\mathcal{E}(1) \rightarrow 2.$$ 

For the local case, we instead take coherent functors

$$P : \mathcal{E} \rightarrow \text{Set}.$$ 

These correspond exactly to Henkin models of (the theory represented by) $\mathcal{E}$.

The topology is given (roughly speaking) by basic open sets of the following form, for all formulas $\varphi$:

$$V_\varphi = \{ P \mid P \models \varphi \}$$
Local sheaf representation for toposes

The structure sheaf $\tilde{\mathcal{E}}$ is first defined as a stack on $\mathcal{E}$ by “slicing”,

$$\tilde{\mathcal{E}}(A) = \mathcal{E}/A.$$

The stack is first strictified to a sheaf, and then transferred from $\mathcal{E}$ to the space $X_\mathcal{E}$ of models using a topos-theoretic covering theorem due to Butz and Moerdijk.

For the global sections $\Gamma$, we then have:

$$\Gamma(\tilde{\mathcal{E}}) \cong \mathcal{E}/1 \cong \mathcal{E}.$$

And for the stalks $\tilde{\mathcal{E}}_P$ we have the colimit,

$$\tilde{\mathcal{E}}_P = \lim_{\text{filtered}} \mathcal{E}/A,$$

where the (filtered!) category of elements $\int P$ of the Henkin model $P$ takes the place of the prime filter.
Boolean algebras and Stone duality

The results for toposes suggest an analogous treatment for pretoposes which would be somewhat better, because the models involved would all be standard ones, rather than Henkin style, non-standard models.

Moreover, we then recognize the analogy to Stone duality for Boolean algebras. From the logical point of view, we have a new Stone duality for first-order logic, with the classical theory as the propositional case.
Recall that for a boolean algebra $B$ we have the Stone space $\text{Spec}(B)$, defined exactly as for the subterminal lattice $\text{Sub}_E(1)$ of a topos $E$ (i.e. the prime spectrum). We can represent the points $p \in \text{Spec}(B)$ as boolean homomorphisms,

$$p : B \longrightarrow 2.$$ 

We can recover $B$ from the space $\text{Spec}(B)$ as the clopen subsets, which are represented by continuous maps,

$$f : \text{Spec}(B) \longrightarrow 2,$$

where 2 is given the discrete topology. (This is just a constant sheaf representation!)
Boolean algebras and Stone duality

There is a contravariant equivalence of categories,

\[
\begin{array}{ccc}
\text{Spec} & \xrightarrow{\sim} & \text{Stone}^{\text{op}} \\
\text{Bool} & \xleftarrow{\sim} & \text{Clop}
\end{array}
\]

The functors are given just by homming into \(2\).

Logically, a Boolean algebra is (the Lindenbaum-Tarski algebra of) a \textit{theory in propositional logic}, and a boolean homomorphism \(B \to 2\) is a \textit{model}, i.e. a truth-valuation.

We shall generalize this situation by replacing Boolean algebras with (Boolean) pretoposes, representing first-order theories, and replacing \(2\)-valued models with \textbf{Set}-valued models.
The generalization to Boolean pretoposes works like this:

<table>
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<tr>
<th>Boolean algebra $B$</th>
<th>Boolean pretopos $\mathcal{B}$</th>
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<tr>
<td>propositional theory</td>
<td>first-order theory</td>
</tr>
<tr>
<td>homomorphism $B \rightarrow 2$</td>
<td>pretopos functor $\mathcal{B} \rightarrow \mathbf{Set}$</td>
</tr>
<tr>
<td>truth-valuation</td>
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</tr>
<tr>
<td>topological space</td>
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<tr>
<td>$\text{Spec}(B)$</td>
<td>$\text{Spec}(\mathcal{B})$</td>
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<tr>
<td>of all valuations</td>
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<td>$\text{Spec}(B) \rightarrow 2$</td>
<td>$\text{Spec}(\mathcal{B}) \rightarrow \mathbf{Set}$</td>
</tr>
<tr>
<td>clopen set</td>
<td>coherent sheaf</td>
</tr>
</tbody>
</table>
Stone duality for pretoposes (Awodey-Forssell)

Theorem (A.-Forssell 2008)

There is a contravariant adjunction,

\[
\begin{array}{ccc}
\text{BPreTop} & \xleftrightarrow{\text{Spec}} & \text{StoneTopGpd}^{\text{op}} \\
\xleftarrow{\text{Coh}} & & \\
\end{array}
\]

in which the functors are given by homming into \textbf{Set}.

The spectrum \textbf{Spec}(\mathcal{B}) of a Boolean pretopos \mathcal{B} is the groupoid of models and isos, topologized by “satisfaction of formulas”.

Recovering \mathcal{B} from \textbf{Spec}(\mathcal{B}) amounts to recovering an elementary theory from its models. This is done by taking the coherent, equivariant sheaves on \textbf{Spec}(\mathcal{B}), using results from topos theory.
Stone duality for pretoposes (Awodey-Forssell)

Makkai has found a related equivalence:

\[ \text{BPreTop} \cong \text{UltraGpd}^{\text{op}} \]

We replace Makkai’s ultraproduct structure on the groupoids of models by a Stone-style logical topology.

For us, however, the “semantic” functor,

\[ \text{Spec} : \text{BPreTop} \longrightarrow \text{StoneTopGpd}^{\text{op}} \]

is not full: there are continuous functors between the groupoids of models that do not come from a “translation of theories”.

Compare the case of rings \( A, B \): a continuous function

\[ f : \text{Spec}(B) \longrightarrow \text{Spec}(A) \]

need not come from a ring homomorphism \( h : A \longrightarrow B \).
Sheaf representation for pretoposes (Awodey-Breiner)

As for rings and affine schemes, we can equip the spectrum $\text{Spec}(\mathcal{B})$ of the pretopos $\mathcal{B}$ with a "structure sheaf" $\tilde{\mathcal{B}}$, defined as in the sheaf representation for toposes:

- start with the presheaf of categories $\tilde{\mathcal{B}} : \mathcal{B}^{\text{op}} \to \text{Cat}$ with,

$$\tilde{\mathcal{B}}(X) \cong \mathcal{B}/X,$$

for all $X \in \mathcal{B}$. This is a stack because $\mathcal{B}$ is a pretopos.

- Strictify to get a sheaf of categories on $\mathcal{B}$.

- Use the equivalence of toposes,

$$\text{Sh}(\mathcal{B}) \simeq \text{Sh}_{\text{eq}}(\text{Spec}(\mathcal{B}))$$

$\mathcal{B}$ is determined so that this equivalence holds.

- Move $\tilde{\mathcal{B}}$ along this equivalence in order to get an equivariant sheaf on $\text{Spec}(\mathcal{B})$. 
The trasported $\tilde{\mathcal{B}}$ is an equivariant sheaf of local, boolean pretoposes on the topological groupoid $\text{Spec}(\mathcal{B})$.

- Logically, $\tilde{\mathcal{B}}$ is a sheaf of “local theories” on the groupoid of models, equipped with the logical topology.
- As before, $\tilde{\mathcal{B}}$ has global sections $\Gamma\tilde{\mathcal{B}} \simeq \mathcal{B}$. So the original “theory” $\mathcal{B}$ is the “theory of all models”.
- The stalk $\tilde{\mathcal{B}}_P$ at a model $P : \mathcal{B} \rightarrow \text{Set}$ is a well-pointed pretopos: it is the “elementary diagram” of the model $P$.
- The global sections functors $\Gamma_P : \tilde{\mathcal{B}}_P \hookrightarrow \text{Set}$ are faithful pretopos morphisms, i.e. “complete models”.
In sum, we have the following:

**Theorem (A.-Breiner 2013)**

Let $\mathcal{B}$ be a boolean pretopos. There is a topological groupoid $G$ with an equivariant sheaf of pretoposes $\tilde{\mathcal{B}}$ such that:

1. for every $x \in G$, the stalk $\tilde{\mathcal{B}}_x$ is a well-pointed pretopos,
2. for the pretopos of global sections, we have: $\Gamma(\tilde{\mathcal{B}}) \cong \mathcal{B}$.

Thus every Boolean pretopos is isomorphic to the global sections of a sheaf of well-pointed pretoposes.
In sum, we have the following:

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Thus every Boolean pretopos is isomorphic to the global sections of a sheaf of well-pointed pretoposes.

There is an analogous result for non-Boolean case, with local pretoposes in place of well-pointed ones in the stalks.
Sheaf representation for pretoposes (Awodey-Breiner)

The resulting sub-direct-product representation $\mathcal{B} \hookrightarrow \prod_x \tilde{\mathcal{B}}_x$ yields:

**Corollary (Gödel completeness theorem)**

There is a pretopos embedding,

$$\mathcal{B} \hookrightarrow \prod_{x \in G} \tilde{\mathcal{B}}_x \hookrightarrow \prod_{x \in G} \text{Set} \simeq \text{Set}^{\lvert G \rvert},$$

with $\lvert G \rvert$ the set of points of the topological groupoid $G = \text{Spec}(\mathcal{B})$ of models.
Sheaf representation for pretoposes (Awodey-Breiner)

The resulting sub-direct-product representation \( \mathcal{B} \rightarrow \prod_x \tilde{\mathcal{B}}_x \) yields:

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There is a pretopos embedding,

\[
\mathcal{B} \rightarrow \prod_{x \in G} \tilde{\mathcal{B}}_x \rightarrow \prod_{x \in G} \text{Set} \cong \text{Set}^{|G|},
\]

with \(|G|\) the set of points of the topological groupoid \( G = \text{Spec}(\mathcal{B}) \) of models.

Other consequences of the sheaf representation include:

- Conceptual completeness (Makkai-Reyes)
- Beth definability (Makkai)
Logical schemes

For a Boolean pretopos $\mathcal{B}$, call the pair

$$(\text{Spec}(\mathcal{B}), \tilde{\mathcal{B}})$$

an affine **logical scheme**.

A morphism of logical schemes

$$(h, \tilde{h}) : (\text{Spec}(\mathcal{B}), \tilde{\mathcal{B}}) \rightarrow (\text{Spec}(\mathcal{A}), \tilde{\mathcal{A}})$$

consists of a continuous groupoid homomorphism

$$h : \text{Spec}(\mathcal{B}) \rightarrow \text{Spec}(\mathcal{A}),$$

together with a pretopos functor

$$\tilde{h} : \tilde{\mathcal{A}} \rightarrow h_* \tilde{\mathcal{B}}$$

over $\text{Spec}(\mathcal{A})$. 
Theorem (A.-Breiner 2012)

Every pretopos functor \( A \rightarrow B \) induces a morphism of the associated affine logical schemes. Moreover, the functor

\[
\text{Spec} : \text{BPreTop} \rightarrow \text{AffLScheme}^{\text{op}}
\]

is full: every map of schemes comes from a map of pretoposes.

Corollary

There is a contravariant equivalence:

\[
\text{BPreTop} \simeq \text{AffLScheme}^{\text{op}}
\]

between Boolean pretoposes and affine logical schemes.
References