A Behavioral Interpretation of Mathematics and Logic

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Why do we care?

• The better we understand the behavioral elements, the better we can engineer the behavior in ourselves or others.
• Mathematics and logic are commonly difficult for people.
• Perhaps we can do better.
Kinds of quantitative behavior

• Non-verbal behavior under control of quantitative variables.
  • Use a lever to move a big rock

• Verbal behavior under control of quantitative variables.

• Rule-governed behavior
  • Contingency-shaped behavior becomes a rule when passed on to others verbally: “Use a lever when moving big rocks.”

• Mathematics: a system of rules for modeling and quantifying nature.
  • Mathematical and logical terms are conventional and must be learned like other verbal behavior.

• Logic is a system of rules defining relationships among verbal statements

• As verbal antecedents they can guide effective non-verbal behavior
What is a “behavioral interpretation?”

- Identification of the reinforcement contingencies that
  - Bring behavior under control of quantitative variables.
  - Likely led to the development of formal rules and models within cultures.
  - Facilitate the teaching of such rules out of context (i.e., in the classroom)
How much math do you need to know? Practical contingencies

- Managing time and distance
- Cooking
- Weighing risks and probabilities
- Handling money
  - How much does it cost? What is its “value”? Which can of beans is the better deal?
  - Do I have enough money? How much change should I get?
  - How much is a 15% discount? What’s a 20% tip?
  - Budgeting: How much can I live on?
  - How much do I need for retirement?
  - What can I expect from an investment?
  - Understanding credit, interest rates, and debt.
  - Paying taxes
- Solving novel problems of everyday life (e.g., splitting the bill, estimating time of arrival, figuring out the score in a ball game.)
How much math do you need to know? Social contingencies

• Advancing through school
• Getting into college
• Getting a job
• Avoiding embarrassment
• Looking smart

• Ubiquity of smart-phones has altered some of these contingencies, both practical and social.
Pathways to mathematical skills

• They can be:
  • Taught in school, or on-line
  • Learned from books
  • Learned by experience
  • Derived from rules
  • Memorized
  • Looked up on the web
  • Figured out by problem solving
    • Bringing new stimuli to bear on one’s own behavior
    • Behave under multiple control of old and new stimuli
    • Repeat in cycles
    • Recognize that a solution has been found
Part II: At the beginning:

- Primordial “mathematics” in non-verbal organisms
  - Presumably both innate and acquired behavior can come under direct control of quantitative features of the physical and social environment.
- Controlling stimuli:
  - Number, mass, distance, time of day, season of year, changes in vegetation, maturing of young, number and size of rival or prey, azimuth.
- Behavior
  - Allocating time: nest building, rearing of young, food gathering, migration
  - Hunting: Distance to prey or predator – when to stalk, when to strike, when to flee
  - Storing resources
  - Allocating resources: Sharing, matching, fighting or fleeing
- Selected behavior tends to be good enough to satisfy natural contingencies (e.g. the waggle dance of bees)
A: Forage at 22 degree angle to the sun.
B: Forage at 45 degree angle to the sun.
But how many different dances can bees discriminate? A circle has an infinite number of diameters, but surely bees can discriminate only a dozen or so.
Analogy: How to find the Penn State Stadium
“Cave man” mathematics

Hunter-gatherers, prior to agriculture and domestication of herds
Verbal, and of modern intelligence
• Preverbal quantification, was shared (more or less) with other species. (cf. Robinson Crusoe)
  • But with relatively less innate quantitative behavior.
  • Skills developed through a history of discrimination training by natural contingencies.
Examples of Controlling stimuli

- Number of gazelles
- Weight of rock
- Length of lever
- Height of tree
- Depth of fall
- Density of fruit
- Distance to goal
- Time of day
- Season of year
- Changes in vegetation
- Maturing of young
- Size of rival or prey
Examples of relevant behavior

• Allocating time: rearing of young, food gathering, migration
• Avoiding falls, strain, injury
• Hunting: Timing and coordination of attack; estimating distance of travel and force of projectiles
• Estimating effort – how much can be carried; how far
• Estimating duration
• Estimating distance
• Anchoring shelters against the wind
• Storing resources
• Judging relative sizes
• Allocating resources: Sharing, matching
• Estimating odds: fighting or fleeing
• Tool use: taking mechanical advantage of levers, handles, wedges, slings, weights.
• The behavior and the controlling variables entered into contingencies of reinforcement. Behavior became sensitive to quantitative aspects of the environment and did not require a “system.”
The emergence of extended measurements
(i.e., that could not be detected in momentary stimuli)
They seem to require correlated markers, verbal rules, or conventions;
record-keeping, perhaps in the form of stories or tallies.

• Time and Duration: Natural cycles - days, months, years; by physical changes:
  maturation of children, aging of adults, growth of hair or beard, height of tree,
  flowering and fruiting of crops, fatigue when walking, sleep, healing of wounds;
  the burning of wood to ashes; legendary accomplishments.

• Distance: Landmarks; relative size of trees and mountains; stories: The water
  hole is going dry, and we must walk beyond the great mountain to the river.

• Abundance: how much firewood can you get from a copse? How many people
  can feed on a mastodon, and for how long?

• Comparison: Which is heavier, a ten-foot pole or a twelve-inch boulder? Is it
  farther to the Great Meadow or the water hole?

• Numerosity beyond “one, two, three, many”: How large is the tribe? How many
  children? How productive is the blueberry bush? How large is the approaching
  force?
Relevant contingencies:
Tacting and quantifying stimuli guided action. “Systems” increase precision and provide more precise control than intuition or judgment.

- “It is a four days walk to the next water hole.”
- “There are 17 warriors approaching.”
- “It will take us a month to move our camp to the top of the bluff by the river.”
- “12 men can drag a mastodon carcass on level ground.”
- “Everyone will get 3 shares.”
Cave man contingencies are still with us

• The refinements of modern education overlie street skills shaped up by the same contingencies of 50,000 years ago.
Kinds of verbal quantification:

• Tacting quantitative variables: (as in previous slides)
• Tacting relationships and comparisons:
  • Tacting temporal relationships such as before, next, after, past, future, and duration
  • Tacting physical relationships such as above, below, adjacent, near, far, heavy, and light. density E.g., lighter than water
• Counting: the quantification of integer variables
• Estimating: the quantification of continuous variables
Verbal quantification (cont’d.)

• Extraction of rules of thumb:
  • levers, wheels, wedges, tool making, building, flotation.

• Transmission of rules across generations: The accumulation of quantitative lore.

• Sharpening of stimulus control by dividing continuous variables into categories (e.g. seasons of the year, morning/afternoon/evening, north/south, etc.)

• Orienting listener to relevant stimuli

• All of the above were shaped up by conventions of a verbal community
  • Depends on local contingencies and historical accident: (e.g., some Amazon tribes do not count above ~5.)
Rules of thumb:
Quantification was approximate

• Reflecting the fact that natural contingencies are crude:
  • If you hunt twice as frequently, you might catch half again as much game, or three times as much. \([H = G; \ 2H = 1\frac{1}{2}G; \ 2H = 3G. \ ????]\)
  • It takes a day to scale the mountain, but half a day to return. \([M = D; \ M = \frac{1}{2}D \ ???? \ ]\)
  • From one bush you can fill a pouch with berries; from a second bush hardly any: 1 bush = 1 pouch; 2 bushes = 1 pouch. \([B=P; \ 2B=P. \ ????]\)
Two reasons quantification was approximate

• Uncontrolled variability
  • Little control over the environment [science had not emerged]
  • Natural phenomena are intrinsically variable

• Deficient technology of measurement
  • Lack of standardization
  • Primitive tools, deficient in:
    • Accuracy
    • Reliability
    • Validity
      • The rotation of the earth is accurate, reliable, and valid, but day length, as a measure, is not.
But even for cave men, two kinds of measurement emerged
Some variables are *continuous*: They change imperceptibly

Growth of a plant, a beard, or a child.

- Movement of sun, moon, stars, and planets.
- Rise of the tides.
- Wearing of pathways.
- Distance to a target

- We usually *estimate* continuous measures.
- He is a giant, the water is rising, it is late in the day, the path is more worn, it will take all afternoon to get there.
Some variables are *discrete*
They jump from one value to the next

• The number of eggs, arrowheads, people, mastodons, or pots.
• They can be quantified by physical symbols, e.g. tick marks, or beads on a thong.
  • Why are beads on a string better than stones in a bowl as counters?
    • (Easier to keep track and to compare, and you don’t need numbers.)
• This many days have elapsed since Thag left for the hunting grounds:
• Discrete variables “jump” from one value to the next.
  • You can have 3 children, or 4 children, but not 3½ children.
  • A chicken laid 5 eggs last week, not 4.75 eggs.
  • (Need not be integer variables: Shoe sizes. Ice cream cones, shirt sizes are discrete. Oranges, walnuts, etc. can separate into natural fractions.)

• In continuous variables there is no identifiable “next” value.
  • If you were once 67 inches tall and are now 68 inches tall, presumably there was a time when you were 67.666 inches tall and 67.6661 inches tall.

• Continuous variables can be represented by a line:

• Discrete variables can be represented by a row of points:

  . . . . . . . . . . . . . . . . . . . . . .
• Tacts of discrete variables are under control of discrete stimuli or of discrete responses to stimuli (e.g. four boys, and/or the intraverbal response “four”)

• Tacts of continuous variables are shaped up by discrimination training:
  • height (short, medium, tall), weight (light, heavy, humongous);
  • time (morning, noon, evening, night, early, late);
  • distance (right here, over there, nearby, far);
  • age (infant, child, lad, young man, man, old man) [Was it a boy or a woman?]
• Responding to continuous variables is necessarily approximate.
  • Discrimination of continuous variables requires consistent contingencies.
  • In the absence of standardized measurements, they were inconsistent.
  • Limitations of sensory organs
  • Natural contingencies are approximate.
  • Classes of behavior do not correspond to classes of stimuli: A pile of sand added to a pile of sand is a big pile of sand. Take away a pailful, and you still have a big pile of sand.

• But responding to discrete variables can be precise: Our classes of behavior can correspond to the classes of stimuli.
  • Contingencies of reinforcement shape up classes of behavior whose boundaries correspond closely to the stimulus classes.

• In short:
  • Continuous changes often can’t be discriminated
  • Discrete changes are usually relatively easily discriminable
• Discrete and continuous are in the contingencies and need not reflect “truths” of nature:
  • (Is nature fundamentally discrete? Quanta, atoms, etc.)
  • Divisions of age, size, ethnicity, eye color, phonemes
  • Sand compacts into sandstone, which erodes into sand, or metamorphoses into quartzite. Bits of clay become mud, compacts into shale, which erodes into bits of clay, or metamorphoses into slate.
  • Cups of water are discrete when you are measuring them into a recipe, but continuous once they are so measured.
  • The legs of a horse are discrete at the ends, but continuous at the shoulders.
Part III: Transition to precise quantification

- The contingencies described so far led to rules of thumb that must have been in effect for many thousands of years.
- Every such act of quantification or estimation must have been biased by a host of conditions in effect at that time or place.
- What contingencies led to the relatively objective verbal rules of logic and mathematics, and how did they arise from the rules of thumb that were shaped by long exposure to such natural contingencies?
Barter

• Sharing, with the expectation of reciprocation, is a primitive form of barter and can be found in non-human animals.

• Specialization of labor must have arisen early
  • Bork is a mighty hunter; Thag makes good pots; Zog is skilled at making arrowheads.

• Specialization developed as cultures evolved, and barter surely kept pace.
Technological advance: The Balance Scale

(Why is it now the symbol for justice?)

https://www.cgtrader.com/3d-models/household/tools/balance-scale
The concept of equality

• Barter is about trading one thing for another thing of “equal” value.
  • Both within and across kinds of things: equality of reinforcers

• People are highly sensitive to “fairness” and being cheated.
  • (The parking meter as metaphor, hotel rooms; sibling rivalry; reputations)

• Equality is a fundamental concept in mathematics: Most of mathematics is about finding what something “equals.”
  • 6x8 equals 48
  • The area of a rectangle equals base times height.
  • 4/5 equals 0.80.
  • (2x – y)² equals (4x² - 4xy + y²)
  • The distance to the sun equals 90 million miles.
  • The tangent of the function (y equals e^x) equals e^x
• Need for precision and standardization
  • Counting can be used for discrete uniform variables
  • The balance scale allows for the quantification of continuous variables and for irregular discrete variables.
• Led to the concept of the variable: $X = Y; \quad X = Z; \quad X = 2W$;
• Profit and loss
• Surplus and Debt, + and -, positive and negative quantities.
• Concept of relative value (conditional on circumstances)

• Concepts of ratio and fractions must have gradually emerged
  • 10 eggs for one flask of oil; 5 eggs for half a flask of oil.
Discovery of a physical principle

Torque = weight times distance from a fulcrum

Archimedes
Permitted precise measurement of fractions and ratios

• To weigh out a quarter pound of grain, hang it four times as far away from the fulcrum as a one pound weight.

• But suppose you want to exchange your cow for oil, flour, fruit, eggs, shoes, a cloak, and a spear?
Money

• A kind of Rosetta Stone that locates all goods on a common scale – a scale representing units of reinforcement.

• You can change your cow into currency and change your currency into oil, flour, fruit, eggs, shoes, a cloak, and a spear at your leisure, and set aside what’s left over for future purchases.

• But it requires a sophisticated quantification: an arithmetic system with equal increments, proportional ratios and with no upper limit (in principle). (Known today as a “ratio scale” or measurement.)

• Apparently arose in Mesopotamia 4000-5000 yrs ago.
From tick marks to Counting

• Ten tick marks take ten times as long to mark as one.
  • And 10 times the effort
  • And requires a medium and a marking tool
  • And might fade too soon or endure too long

• Counting was more efficient, less effortful, and required no medium. But it was meaningless without the appropriate history.
  • Hence, the first math class!
“Counting” as pure intraverbal

• Saying “four” in response to “one, two, three...” is an intraverbal response. The speaker provides the stimulus, you respond.

• Saying “One, two, three, four” etc....” is an intraverbal chain. Your own behavior provides the stimulus for the next response.

• Intraverbal responses and intraverbal chains are not “counting” in a quantitative sense. They merely establish unique markers that have a one-to-one relationship to any set of discrete items.
  • An alphabet could work as well: A-Z, AA-ZZ, AAA-ZZZ, etc.
  • They are analogous to tick marks on a jail house wall:
• “hickory, dickory, dock...” apparently originally used by shepherds for counting sheep, now used in children’s games for counting out players in games.

• This merely reminds us that any intraverbal chain is the same as any other as a counting tool. They are not “like” numbers; they are numbers.

• It’s the intraverbal order that provides the necessary connection with quantity, the one-to-one relationship of ordinal position to number of items.
Counting items is much more complex. Multiple control of tact, intraverbal, and self-echoic

• Line up your tomatoes.
• Begin at one end.
• Point to the “first” (e.g., left-most) item.
• Say “One” under control of the first item. (Tact)
• Move your finger to the right one place (rule)
• Say “Two” under control of that operation and the verbal stimulus “One.” (Tact and intraverbal)
• Move your finger to the right one place.
• Say “Three” under control of that operation and the verbal stimulus “Two.” (Tact and intraverbal).
• Continue....
• When you reach the last item, and have emitted an intraverbal response, announce “There are (N) tomatoes,” where “N” is under control of that circumstance and the prior verbal response. (Tact & self-echoic)
• But we don’t usually line them up, and we don’t often point.
  • Rather, we orient from one to the next in a systematic way: a learned strategy
  • That is, we learn strategies for keeping track of items already counted, e.g., by putting items aside as we count. Put your apples in a bag.
• It’s difficult when things are in motion, or are clustered, or can’t easily be separated:
  • people attending a dance (or an inauguration)
  • Beans in a jar
  • Honey bees on a comb
Successful counting requires discriminating the counted from the uncounted, i.e., strategic behavior:

- Marking
- Segregating
- Arranging
- Moving systematically
- Delegating and adding
• For centuries, the concept of number was represented crudely (e.g., Roman numerals, Babylonian cuneiform).
• The modern number system arose in India (along with the concept of zero) around 500 AD and soon spread to China and the Middle East.
• The Arabs made great advances in the development of formal mathematics, including al-gebra.
  • Compare: al-cohol, al-gorithm, al-kali, al-chemy
• It did not reach Europe until ~1100 AD.
• Whether clumsy or efficient, a number system was required by the use of money as a commercial medium.

• Unlike goods, the unit of money was “pure.”

• One radish does not equal one radish, but one penny equals exactly one penny, regardless of incidental differences in the coins.

• That is, numbers are *abstractions* rather than things.

• The number system is insulated from circumstances, age, weather, condition. One remains one, whether whispered, shouted, covertly counted, or chiseled in stone.
• Numbers, whether “4,” “four,” or “IV,” are verbal responses
• Recall Skinner’s definition of verbal behavior: A verbal response is shaped up by arbitrary contingencies maintained by verbal communities.
• The control of the behavior of the listener is a convention and doesn’t depend on trivial variations in the stimulus. Verbal contingencies (generally) do not distinguish between “FOUR” and “four”
• So “one” remains “one,” so long as it can be discriminated as such by the verbal community.
It is the arbitrary nature of verbal contingencies that makes numbers “pure.”

- You can count indefinitely, without having anything to count.
- You can add and subtract, multiply and divide, without having anything to add or subtract: \[412-169 = 243\], \[2.5\% \text{ of } 88 = 2.2\].
- You can play with numbers without asking whether they mean anything:
  - Fermat: Is \(2^N + 1\) always a prime number?
  - Is \(\sqrt{2}\) “irrational?”
    - [That is, can it be represented as the ratio of two whole numbers? In 500BC(!), Pythagoras said “no.” \(\sqrt{2}\) is neither even nor odd, no matter how far you move the decimal place.]
How verbal behavior allows abstraction

• The verbal response “fish” can be occasioned by a zillion unique stimuli.
• The auditory stimulus “fish” collapses all of that variability into a single stimulus.
• The listener cannot recapture any of the original variability.
Fish!

...and chips
“The listener cannot recapture any of the original variability.”

“FISH!”
• There is likely to be variability in the listener’s response from one occasion to the next, but it will be controlled by idiosyncratic and incidental variables. It is cut off from the variables that control the behavior of the speaker.
Numbers become abstract for the same reason

“Four” is a response common to an unlimited number of sets of four things.

Skinner asserts that abstraction is purely a verbal process, because only a verbal community can bring a response under control of a single property of a stimulus.

But to continue our speculations about the evolution of quantitative behavior....
The Rise of Agriculture

• The rise of agriculture encouraged permanent settlements
• Permanent settlements led to the quantification of area.
• Both entailed a host of new contingencies of reinforcement.
Examples

• The contingencies of agriculture required the development of 2-dimensional geometry of area.
• The storage of grain, flour, and other goods required a 3-dimensional geometry of volume.
• The contingencies of making roads required the development of the geometry of lines and angles.
• Large settlements required division of land
• Animal husbandry entailed multiplication. (“Go forth and multiply”)
• Building trades required the quantification of strength of materials.
Allocation of Land:
Ownership, socage, serfdom, family plots, etc.

• When land became a commodity to be bought, sold, traded, or inherited, systematic division and quantification were required.

• Natural boundaries are simple, but areas are hard to quantify.

• Circles were simple to make. Tether a goat to a stake, and soon you have a circle. But they are wasteful.

• Rectangles can use all the available space, but they are hard to make accurately without a compass as a surveying tool, which came much later (~1000 AD)
  ....unless you had a mastery of geometry.
Geometry

• Early units of distance and area were pragmatic and approximate:
  • Acre
  • Furlong
  • Foot
  • Hand
  • Fathom
  • Mile

• Contingencies favored units that Farmer Brown could employ.

• But inheritance, barter, and sale required more precise quantification of area.
“Essentialistic” forms, 2-D and 3-D
Leap to a purely formal system: Abstraction

• Like number, geometric shapes are purely abstract verbal behavior.
• In nature, there are no circles, points, or squares, no diamonds, hexagons, or dodecahedrons.
• But what about garnet crystals or salt crystals? Ripples on a pond? Pasture for a tethered goat? Honeycomb? Spider webs? The eye of the dragonfly?
• Of course the approximations are usually “good enough” for many practical purposes. The point is that the abstraction is “perfect” and nature is not.
• There is a fundamental mismatch between our models and the world being modelled.
Geometry is a formal system, an abstract system, not a physical one:
  • Postulates (axioms) and deductions

It generates precise models, about which perfect deductions can be made. Those models do not exactly match nature, but they are useful in making predictions and guiding action.

Example: The 2-mile addition. What is the edge of a river? How do you run a straight line, exactly 6 miles long through swamps and mountain laurel thickets, up and down ravines, exactly parallel to another line 9 miles to the south?

Nathaniel Maynard’s lot was laid out to be 100 rods by 160 rods or exactly 100 acres. It is now known to be about 109 acres. (Note the “about.” The added precision of the 21st century merely reduces the error; it does not eliminate it.)
Use of symbols to represent variables

• Let $a =$ the short side of a rectangle and $b$ the long. Area = $a \times b$
• Area of a triangle = $\frac{1}{2}$ base $\times$ height
• Volume of a sphere = $\frac{4}{3}\pi r^3$, where $\pi =$ the ratio of circumference to diameter of a circle.
The roots of algebra

• The square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides: $a^2 + b^2 = c^2$

• The diagonal of a square = $\sqrt{2}a^2$
If a line is “a” units long, what is “a times a”? Let a = 6.

\[ a \times a = a^2 = 36 \]

Multiplication is just addition. To multiply something by, say, 6, just means to add that something to itself 6 times. If we add “a” to itself “a” times, we can make a perfect square, or “a^2”.
If one line is “a” units long, and a second line is “b” units long, what is a times b?
Let a = 6 and b = 3

\[ a = 6 \]
\[ b = 3 \]

\[ ab = 18 \]
How much bigger will your plot be if you make it 2 feet wider and 2 feet longer?
What is $(a+b)^2$? (Let $a=6$ and $b=2$)

$$(a + b)^2 = a^2 + 2ab + b^2$$
What is \((a-b)^2\)? (Let \(a=8\) and \(b=2\))

\[
(a - b)^2 = a^2 - 2ab + b^2
\]
Area of a rectangle = base times height = ab

\[ a \]

\[ b \]

Area of a right triangle = \( \frac{1}{2} \) base times height = \( \frac{1}{2}ab \)
Area of a triangle = $\frac{1}{2}$ base times height = $\frac{1}{2}ab$

Pythagorean theorem: $a^2 + b^2 = c^2$
\( c^2 = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + (a-b)^2 = 2ab + a^2 - 2ab + b^2 = a^2 + b^2 \)
• Much of this was figured out before arithmetic rules, as we know them, were developed.

• Pythagoras preceded the concept of “zero” by a thousand years.

• The Greeks (and others) could calculate, but not using the same system we use.
Early Science and Engineering:
The exploitation of mathematics and
the emergence of new contingencies

- Roads
- Aquaducts
- Walls
- Terraces
- Homes, storehouses, and palaces
- Burial monuments (Pyramids)
- Astronomical clocks (Stonehenge, Mayan temples)
- Statues & Monuments
- Ship-building
- Navigation
- Weaponry
Algebra

- Algebra is the use of symbols for simplifying mathematical operations.
- An equation written in symbols can take an unlimited number of exemplars.
- Circumference of circle = \( \pi D \), for any value of \( D \).
- \((a+b)^2 = a^2 + 2ab + b^2\) This holds true for any values of \( a \) and \( b \).

- Binomial formula
  \((a+b)^N = [N!/N!0!]a^Nb^0 + N!/(N-1)!1!]a^{N-1}b^1 + ...... [N!/N-N)!0!]a^0b^N\)
  - Fundamentally, it’s just a bunch of numbers added together
  - But it can be generalized to an unlimited number of examples, and some are very useful.

- In each case, the algebraic equation is a sentence with variable terms in place of some of the words.
Verbal analog

• The intraverbal frame (aka the “autoclitic frame”)
• Verbal strings with some fixed elements and some variable elements. The fixed elements come to strength in certain contexts, and the variable elements are provided by the context:
  • The X gave the Y to the Z (The lawyer gave the subpoena to the defendant.)
  • X promised Y that Z (Larry promised Curly that he’d mow the lawn.)
  • On the X (On the table; on the shelf; on the top of the fridge)
  • X is Y (He is a liar; Sam is a librarian; my advisor is a behavior analyst.)
• Facility with intraverbal frames arises early in life and becomes fluent and effortless and under good stimulus control.

• The ease with which we plug in variable terms to such frames surely facilitates the substitution of terms in algebraic “sentences.”
Two types of verbal definitions

• Empirical definitions

• A priori definitions

• An empirical definition is “discovered.” You have to observe nature and make a generalization or an abstraction. Then you refine it as more data come in. E.g., How do you define “coyote”?

• A priori definitions are asserted from one’s armchair for the sake of discussion. E.g., a Democrat could be defined as someone who is registered with the local town hall as a member of that party, regardless of who they vote for or what positions they hold.
• Mathematics as a system of a priori definitions, postulates, and deductions.
• A circle defined as “a set of points on a plane equidistant from a single (central) point.”
• A triangle defined as “a closed plane figure with three straight sides.”
• It follows that the sum of the angles in a triangle must be 180 degrees.
• So defined, there are no circles in nature.
Rise of formal or “pure” mathematics

• Departure from approximate mathematics.
• There are no circles or triangles in nature.
Example of models vs. reality.

• Andrew Gardner bought 12 acres of land in 1793, and its boundaries were straight lines. Perfect and pure straight lines.

• He farmed his land and marked his boundaries.
Here are some of his boundary walls
Some of the stones weigh hundreds of pounds and must have been moved into place by oxen.
Take-home message:

• The concepts of math and geometry are abstractions.
• They aren’t “real.”
• They are verbal constructions.

• Does 1+1 = 2? Only in the world of our assumptions.
• Add a cup of water to a cup of alcohol and get 1.9 cups of liquid.
• Add a foot of snow to a foot of snow and get 18 inches of snow.
Mathematical tools are *models* of nature. Hence they depart from reality.

- Our models start with definitions and postulates, e.g. that certain variables are continuous;
- Nature might or might not be continuous, but we cannot discriminate the differences between natural phenomena at that level, so our model might be good enough.
- Sometimes we use continuous models of discrete variables, because although they are wrong, they are easy to use and are good enough for many purposes.
  - Examples: statistics, the normal curve.
The role of assumptions in mathematics

• Statistical procedures are mathematical edifices that depend on a set of assumptions about how the world works.

• The t test: assumes normally distributed variables, random sampling, and equal variances in the two populations.
Formula for the normal curve

\[ Y = \frac{e^{-(X - \mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma^2} \]
The normal distribution for height of men
Example of using the normal curve as a model

• Sergeant Bixby is in charge of supplying uniforms for new army recruits, but he is all out of size Medium male uniforms. Men who are between 67 and 72 inches tall wear a size Medium. A cohort of 3,000 recruits is due to arrive next month. How many new Medium uniforms should Sergeant Bixby order?
Example of using a mathematical model of nature
Example of using a mathematical model of nature

• The curve is supposed to represent the distribution of heights of men in the entire population.

• The proportion of the area under the normal curve, between 67 and 72 inches (the shaded area) is taken to be the proportion of the population of men who are between 67 and 72 inches tall.

• Using the formula for the normal curve given above, it is possible to figure out that area using the techniques of calculus, or equivalent tables. It is 0.5889 or 58.89% of the total area.

• So 58.89% of Bixby’s recruits will need a size medium uniform.

• 3,000 x .5889 = 1,766.7

• So Bixbly should put in an order for 1,766.7 size Medium uniforms.
The precision of the answer is absurd. Height isn’t “really” normally distributed, and no set of 3000 recruits will be perfectly representative.

- A better answer would be “around 2000.”
- There are good reasons for being skeptical about the use of statistics to make decisions about experimental results.
- But the model provides an algorithm for answering the question, and we can take it as a starting point, a guidepost.

- Of course, some models are better than others.
Another example with discrete variables:

• A female mosquito can lay around 2000 eggs in its lifetime. If all of the offspring of a male and female mosquito lived, how thick on the ground would the offspring be after 10 generations? Note that the circumference of the earth is 25,000 miles at the equator.

• Procedure:
  • Figure out the surface area of the earth in square inches
  • Figure out how many mosquitoes there would be after 10 generations
  • Divide the latter by the former to find #mosquitoes/square inch.
Area of earth in square inches

- Area of a sphere = $4\pi r^2$
  (The area of a sphere is exactly four times the area of any of its great circles. Why is the ratio an integer? These are idealized forms. There are neither circles nor spheres in nature. So the area of a cut hemisphere must be $3\pi r^2$.)

- $\pi = 3.14159$
- $r = 4,000$ miles
- 1 mile = 5,280 feet
- 1 foot = 12 inches
- Radius of earth in inches = $(4,000)(5,280)(12) = 253,440,000$
- Area of the earth in square inches = $(4)(3.14159)(253,440,000)^2 = 800,000,000,000,000,000$ square inches.
Number of mosquitoes after 10 generations

• 1 1,000
• 2 1,000,000
• 3 1,000,000,000
• 4 1,000,000,000,000
• 5 1,000,000,000,000,000
• 6 1,000,000,000,000,000,000
• 7 1,000,000,000,000,000,000,000
• 8 1,000,000,000,000,000,000,000,000
• 9 1,000,000,000,000,000,000,000,000,000
• 10 1,000,000,000,000,000,000,000,000,000,000
• Times 2 = 2,000,000,000,000,000,000,000,000,000,000
2,000,000,000,000,000, 000,000,000,000,000 mosquitoes

\[
\frac{2,000,000,000,000,000,000,000,000,000,000,000}{800,000,000,000,000,000,000,000,000,000,000,000}
\]

= 2.5 trillion mosquitoes per square inch
Despite all the idealizations and assumptions

The point is that the spurious accuracy of our answer is irrelevant. It doesn’t matter if a mosquito lays 2000 eggs, or 200 eggs, or 3 eggs, or whether the radius of the earth is 4000 miles or 5280 miles, or whether the earth is a perfect sphere or an oblate spheroid. We can see the general truth emerge: There must be dramatic limits on the reproduction of organisms. There must be a “struggle for survival.”

• It was such considerations that informed Malthus and Darwin, and abstract mathematics gave the argument force, despite its flagrant idealizations.
Concluding remarks about mathematics
Part IV: Logic
Skinner on inference

• When a speaker accurately reports, identifies, or describes a given state of affairs, he increases the likelihood that the listener will act successfully with respect to it, and when the listener looks to the speaker for an extension of his own sensory capacities, or for contact with distant events, or for an accurate characterization of a puzzling situation, the speaker's behavior is most useful to him if the environmental control has not been disturbed by other variables. This is the distinction between fact and fancy, truth and fiction. Similarly, when a speaker intraverbally reconstructs directions, rules of conduct, and "laws of thought," he increases the likelihood of successful practical, ethical, and intellectual behavior, respectively, and his success in doing so depends upon the "purity" of the controlling relations. (VB p. 418)
Skinner Cont’d.

• If we have put something in one of two boxes labeled A and B and as the result of looking in B we say *It is not in B*, we can also construct the response *It is in A*. This has the form of a complex tact, such as might be emitted after looking in A, but it is reached by construction. (VB p. 426)
Propositions conceived as verbal behavior

• It makes no sense to say that “behavior is true” or “behavior is false.”
• Behavior is controlled by its antecedents and consequences. Truth is not intrinsic to the behavior; it is a tact evoked in an observer.
• We say a statement is “true” if action normally controlled by that statement leads to a reinforcing outcome:
• In everyday logic, “There is milk in the fridge” is said to be true if it leads to effective action.
• In formal logic, truth is a property of statements according to a system of rules. An assignment of truth value can only be done after the statement is translated into logical terms.
Everyday Logic

• Given the current state of affairs, what will happen next?
  • A bird is building a nest.
  • A man walks into a hardware store.
  • A police car pulls in behind a truck and puts on his lights.

• Given the current state of affairs, what must have happened before?
  • A flat rock is perfectly balanced on a smaller rock.
  • A watercolor painting is lying on a path in the woods and the colors have not run.

• In our experience, X implies Y.
  • With different experiences, X would imply Z, or W.

• Everyday deductions are a matter of experience and will vary somewhat from person to person.
Formal Logic

• An idealize system of rules like a mathematical model or Euclid’s geometry.

• All dogs have four legs
• Fido is a dog
• Fido has four legs

• If we accept the premise as “true” then Fido **must** have four legs.
• If Fido does not have four legs, we insist that the premise must have been false.
Formal Logic

• Commonly concerned with rules for
  • Substitution of terms
    • “All mice are mammals” is not derived from experience but from rules of categorization.
  • Synonymy
  • Mutual exclusivity
  • Disjunction
  • If – then relationships
  • Deduction

• Expressions like “all” have a prescriptive and an incommensurate descriptive definition and rules of inference.
### Table of logical equivalents

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula 1</th>
<th>Formula 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commutative</td>
<td>$p \land q \iff q \land p$</td>
<td>$p \lor q \iff q \lor p$</td>
</tr>
<tr>
<td>Associative</td>
<td>$(p \land q) \land r \iff p \land (q \land r)$</td>
<td>$(p \lor q) \lor r \iff p \lor (q \lor r)$</td>
</tr>
<tr>
<td>Distributive</td>
<td>$p \land (q \lor r) \iff (p \land q) \lor (p \land r)$</td>
<td>$p \lor (q \land r) \iff (p \lor q) \land (p \lor r)$</td>
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<tr>
<td>Identity</td>
<td>$p \land T \iff p$</td>
<td>$p \lor F \iff p$</td>
</tr>
<tr>
<td>Negation</td>
<td>$p \lor \sim p \iff T$</td>
<td>$p \land \sim p \iff F$</td>
</tr>
<tr>
<td>Double Negative</td>
<td>$\sim (\sim p) \iff p$</td>
<td>$p \lor p \iff p$</td>
</tr>
<tr>
<td>Idempotent</td>
<td>$p \land p \iff p$</td>
<td>$p \lor p \iff p$</td>
</tr>
<tr>
<td>Universal Bound</td>
<td>$p \lor T \iff T$</td>
<td>$p \land F \iff F$</td>
</tr>
<tr>
<td>De Morgan’s</td>
<td>$\sim (p \land q) \iff (\sim p) \lor (\sim q)$</td>
<td>$\sim (p \lor q) \iff (\sim p) \land (\sim q)$</td>
</tr>
<tr>
<td>Absorption</td>
<td>$p \lor (p \land q) \iff p$</td>
<td>$p \land (p \lor q) \iff p$</td>
</tr>
<tr>
<td>Conditional</td>
<td>$(p \Rightarrow q) \iff (\sim p \lor q)$</td>
<td>$\sim (p \Rightarrow q) \iff (p \land \sim q)$</td>
</tr>
</tbody>
</table>
## Rules of inference

<table>
<thead>
<tr>
<th></th>
<th>Modus Ponens $p \implies q$</th>
<th>Modus Tollens $p \implies q$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p$</td>
<td>$\sim q$</td>
</tr>
<tr>
<td></td>
<td>$\therefore q$</td>
<td>$\therefore \sim p$</td>
</tr>
<tr>
<td>Elimination $p \lor q$</td>
<td>$\sim q$</td>
<td>Transitivity $p \implies q$</td>
</tr>
<tr>
<td></td>
<td>$\therefore p$</td>
<td>$q \implies r$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\therefore p \implies r$</td>
</tr>
<tr>
<td>Generalization $p \implies p \lor q$</td>
<td>$q \implies p \lor q$</td>
<td>Specialization $p \land q \implies p$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$p \land q \implies q$</td>
</tr>
<tr>
<td>Conjunction $p$</td>
<td>$q$</td>
<td>Contradiction Rule $\sim p \implies F$</td>
</tr>
<tr>
<td></td>
<td>$\therefore p \land q$</td>
<td>$\therefore p$</td>
</tr>
</tbody>
</table>

Formal logic is analogous to formal mathematical models, like Euclid’s geometry

• Its principles follow whether they make any sense at all.

• All dogs have four legs.
  • Suppose Fido, following an injury, has only three legs.
    • Fido is not a dog.

• All swans are white.
  • Suppose a swan gets covered with gunk from an oil spill and is black.
    • It is not a swan.

• The conclusions are “true” according to the laws of logic, just as it is “true” that some men are less than zero inches tall according to the normal curve.
Rule-governed behavior

• Substitution of terms, symbols.
• Return to the topic of abstraction.
• Rules can be derived descriptively but applied prescriptively.
  • You determine, by observation, that you get 30 miles to the gallon of gas in your car.
  • Divide the distance you plan to drive, in miles, by 30 to find the number of gallons of gas you’ll need. Multiply that number by $3.65 to find the cost of the fuel for your trip.
Skinner on scientific “truth”

• An important part of scientific practice is the evaluation of the probability that a verbal response is "right" or "true"—that it may be acted upon successfully ... Constructed responses are not always fully confirmed, extended tacts are controlled by deviant stimuli, responses to poorly defined or poorly sampled classes of events suffer corresponding disadvantages, generalized reinforcement minimizes but never wholly destroys the effect of the momentary condition of the speaker, and so on. (VB p. 428-429)
The product of the manipulation of terms is usually a textual stimulus (a new equation, for example, or a new form of an expression) which may then lead to other behavior. Sometimes the new expression "solves a problem," sometimes it corresponds with an earlier statement of an hypothesis or theory (this result may be indicated with the autoclitic Q.E.D.), and sometimes the constructed behavior simply leads to effective, possibly nonverbal, action. It is part of the empirical discovery of the logical and scientific verbal community that behavior arrived at in this fashion may be reacted to as if it were a tact or intraverbal response, or some larger sample of the same nature. The behavior of reacting to it in such a way must also be conditioned by the community. (VB p. 424)
On “truth”

• Mathematical “truths” are properties of nature
  • They are independent of the verbal community
    • A plot of flat land that is twice as wide and twice as long as another can grow four times as much grain.
    • “Truth” = Whether it satisfies relevant contingencies of reinforcement, whether you can act on it; whether its practical implications

• Logical “truths” are defined by the rules of the game and need not be reflected in nature. (Includes non-practical mathematics)
  • All swans are white
  • Cygnus is a swan
  • Therefore Cygnus is white.
Concluding remarks about logic and mathematics
Part V: Some tips for teaching mathematics gleaned from my own experience

- NEVER punish, humiliate, or embarrass students, directly or implicitly, for errors in math. Avoid competition among students.
- Punishment, failure, and embarrassment lead to “math anxiety.”
- How many of you would say you have a “math phobia”?
- Among other considerations, conditioned emotional responses interfere with learning.
Keep in mind the relevant verbal operant

- Intraverbal responses are controlled by *specific* antecedent verbal stimuli:
  - “87 years ago…”
  - “Four score and seven years ago” ..... 
  - “Should I commit suicide or not?”....
  - “To be or not to be,...
  - “It’s morning; is it light enough for you to you see....”
  - “Oh say can you see, by the dawn’s early light.....”
  - “Hey George, ....”
  - “Hey Jude, ....”
Consequently behavior will be weakened when the antecedent is different. So teach math facts in different, but equivalent, forms:

- $6 \times 8 = 48$
- $8 \times 6 = 48$
- 6 eights is 48
- 8 sixes is 48
- 6 times what = 48
- 48 = 6 times eight
- Etc.
- Same with covert textual behavior.

- $8 \times 6$
  \[
  \begin{array}{c c c c c}
  X & 6 \\
  \times & X & 8 \\
  \hline
  4 & 8 & 4 & 8 \\
  \end{array}
  \]

- The automaticity of math facts is a good thing, not a bad thing.
Overall approach

• Specify your objectives (outcomes) in terms of behavior: Intraverbals, intraverbal chains, intraverbal frames, textual behavior, transcriptive behavior; rule-governed sequences of behavior.

• Task analysis: fine-grained elements of behavior

• *Ensure mastery of component skills before advancing to composite skills

• *Teach beyond 100% accuracy: Teach to fluency

• Individualize the pace whenever possible

• Make terms concrete: with both examples and non-examples of concepts

• Practical problems that are staged in difficulty
Operationalize objectives

- Students will be able to compute the standard deviation of a set of 5 numbers to one decimal place with a pencil and paper, when the mean is an integer and the numbers are presented as textual stimuli.

- Example: Find the standard deviation of the following numbers to one decimal place:
  
  10  
  14  
  13  
  5  
  8
Identify component tasks in detail

• Recite the formula for a standard deviation (intraverbal chain)
• Identify the terms in the formula (textual)
• Add a list of numbers of any size and write the result (rule-governed behavior; textual behavior, intraverbal behavior, transcriptive behavior)
• Count the numbers in a list (sequence of orienting and intraverbal responses)
• Divide one number by another (algorithm: systematic sequence of intraverbals)
• Subtract a common term from each score in a list and write answers (etc.)
• Multiply two numbers
• Find the difference between two numbers to one decimal place.
• Estimate the square root of a number to one decimal place by a process of successive narrowing.
Teach beyond mastery: Teach to fluency

• This statement is nonsense, even to students with 100% mastery of terms:
  • E.g., “Big Boy tomatoes are normally distributed with a mean of 12 oz. and a standard deviation of 2 oz., whereas Peeping Tom tomatoes have a mean of 10 oz. and a standard deviation of 3 oz. but when both ses of scores are transformed into standard scores, they are both normally distributed with a mean of 0 and a standard deviation of 1.0”

• But not to someone fluent in the vocabulary.

• [Carl Binder experiment]
Ensure mastery (ideally fluency) of component skills before advancing to composite skills.

• In order to do so, individualize the pace, as far as possible.
  • If impossible, have students work in heterogeneous groups part of the time:
    • It can help both the advanced and struggling students, so long as no stigma is attached to the latter.

• Almost nothing in math is difficult if you are fluent in component skills.
• Provide concrete examples
• Provide practical problems that are staged in difficulty
• Provide both examples and non-examples of concepts.
  • (E.g., binomial – sampling with replacement)
Some problem solving strategies

• Role of guessing (estimating) the answer before problem solving
• Blind variation
• Transform terms
• Express as a formula
• Ask a related question
• Transform problem into extreme terms (e.g., either very large or very small numbers)
• Draw a diagram or picture (visual aids)
• Test answer
• (Digression: What is problem solving?)
Final concluding remarks
(Homily on statistical significance?)